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EXTENDING KOTZIG'S THEOREM

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ABSTRACT

The weight of a graph G is the minimum sum of the two degrees of the end points of edges of G. Kotzig proved that every graph triangulating the sphere has weight at most 13, and Grünbaum and Shephard proved that every graph triangulating the torus has weight at most 15. We extend these results for graphs, multigraphs and pseudographs "triangulating" the sphere with g handles S_g, $g \geqq 1$, showing that the corresponding weights are at most about $\sqrt{48g}$, $8g+7$ and $24g-9$, respectively; if a (multi, pseudo) graph triangulates S_g and it is big enough, then its weight is at most 15.

A. Kotzig's Theorem ([6], see also [7]), popularized in the West by B. Grünbaum ([2] and [3]), states that every graph which triangulates the sphere contains an edge in which the sum of the degrees of its end vertices is at most 13; 13 is best lower bound, as can be seen in the Kleetope (see [1] for definition) over the icosahedron, see [6], [2] and [3]. This theorem has been strengthened by E. Jucovič [5] and extended by A. Kotzig [7].

Recently, B. Grünbaum and G. Shephard [4] proved that every graph triangulating the torus contains an edge in which the sum of the degrees of its end vertices is at most 15, and 15 is best possible lower bound (for a related remark, see [2]). They also gave examples of multigraphs (having multiple edges but no loops) dissecting the closed 2-manifold S_g of genus g into triangles, $g \geqq 1$, such that the minimum of all the sums of degrees of end vertices of edges is $13+2g$; they conjectured that every multigraph dissecting S_g into triangles has an edge in which the sum of the degrees of its end vertices is at most $13+2g$, for all $g \geqq 1$.

Let G_g denote the family of all the graphs, containing no loops or multiple edges, which triangulate S_g; each member of G_g induces a decomposition of S_g into a cell-complex, where the closure of each 2-cell is homeomorphic to a closed disc, and every 2-cell meets along edges (1-cells) precisely three other 2-cells, $g \geqq 1$.

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Let MG_g denote the family of multigraphs, containing possibly multiple edges but containing no loops, which triangulate S_g in the sense of [4], i.e., each member of MG_g induces a dissection of S_g into cells, such that the closure of each 2-cell is homeomorphic to a closed disc and each 2-cell meets along edges precisely three other 2-cells; as multiple edges are allowed, two 2-cells need not meet in a connected set, but the meet is always a collection of edges and vertices; in fact, an edge might meet a 2-cell in just its two vertices, and two 2-cells can meet in only their three common vertices, as will be constructed later in the paper (compare [4]).

Let PG_g denote the family of all the pseudographs, containing possibly multiple edges or loops, which dissect S_g into regions such that each region has an interior homeomorphic to an open disc, and every region meets three regions (among which it might meet itself too) along three edges; these edges might be loops.

Let the *weight* $w(E)$ of an edge E in a multigraph G be defined as the sum of the degrees of its end vertices; and if G is a pseudograph and E is a loop, then $w(E)$ is defined as twice the degree of its unique end vertex. Let $w(G)$ be defined as $\min\{w(E) \mid E \in G\}$, and if $\mathcal{G}$ is a family of pseudographs, then $w(\mathcal{G})$ is defined by $w(\mathcal{G}) = \max\{W(G) \mid G \in \mathcal{G}\}$.

In these notations, Kotzig's Theorem states that $w(G_0) = 13$, and Grünbaum–Shephard's Theorem is that $w(MG_1) = 15$ and their conjecture reads "$w(MG_g) = 13 + 2g$ for all $g, g \geqq 2$".

For each integer g define $n(g)$ as the least odd integer which is greater than $6 + \sqrt{48g + 1}$.

The purpose of this paper is to show that $w(G_g) \leqq n(g)$, that $w(MG_g) = 8g + 7$ and that $w(PG_g) = 24g - 9$, for all $g \geqq 1$.

Starting with graphs which triangulate S_g, $g \geqq 1$, we have the following

THEOREM 1. *For every $g, g \geqq 1$, $w(G_g) \leqq n(g)$.*

THEOREM 2. *If $g \geqq 2$ and $48g + 1$ is a complete square, then there is a unique graph G in G_g such that $w(G) = n(g)$; G is the Kleetope over the triangulation of S_g by the complete graph $K_{(n-1)/2}$ on $(n-1)/2$ vertices.*

THEOREM 3. *If $g \geqq 2$ and a graph G in G_g is such that $w(G) = n(g)$, then the maximum degree of G is at most $n + 1$.*

PROOF OF THEOREM 1. Let n denote an odd integer, $n \geqq 11$, and let g be any integer, $g \geqq 0$. Let G be a graph in G_g and suppose $w(G) \geqq n$. Let $e_{i,j}$ denote the number of edges in G having end vertices which are i-valent and j-valent; thus

$e_{i,j} = 0$ holds for all i and j satisfying $i + j < n$. We will show that for a proper choice of n, $w(G) = n$.

Let v_k denote the number of k-valent vertices of G; it is well known (see [1]) that Euler's formula implies

$$3v_3 + 2v_4 + v_5 = 12(1-g) + \sum_{k \geqq 7} (k-6)v_k. \tag{1}$$

If any k-valent vertex V of G would have more than $[k/2]$ neighboring vertices having valences $< n/2$, then some two consecutive (= neighbors) vertices W_1 and W_2 will have valences $< n/2$, so $w(W_1W_2) < n$, contradicting the assumption on G; therefore the following holds:

(2) at most $[k/2]$ neighbors of every k-valent vertex have valences $\leqq (n-1)/2$.

Every k-valent vertex in a triangulation has k different neighbors, therefore

(3) at least $k - [k/2]$ neighbors of every k-valent vertex are of valences $\geqq (n+1)/2$.

It follows from (2) that at most $[k/2]$ neighbors of every k-valent vertex are of valence 3, therefore

$$e_{3,k} \leqq \left[\frac{k}{2}\right] v_k \qquad \text{for all } k, \tag{4}$$

and by summing we get

$$\sum_{k \geqq n-2} e_{3,k} \leqq \sum_{k \geqq n-2} \left[\frac{k}{2}\right] v_k. \tag{5}$$

As $\Sigma_{k \geqq 3} e_{3,k}$ counts all the edges having a 3-valent end vertex, and since $e_{3,k} = 0$ holds for all k, $k \leqq n-4$, it follows that

$$3v_3 - e_{3,n-3} = \sum_{k \geqq n-2} e_{3,k} \leqq \sum_{k \geqq n-2} \left[\frac{k}{2}\right] v_k. \tag{6}$$

Considering all the 3-valent and 4-valent vertices which are neighbors to a k-valent vertex, it follows by (2) that

$$e_{3,k} + e_{4,k} \leqq \left[\frac{k}{2}\right] v_k \qquad \text{for all } k, \tag{7}$$

hence

$$3v_3 + 4v_4 - e_{4,n-4} = \sum_{k \geqq n-3} (e_{3,k} + e_{4,k}) \leqq \sum_{k \geqq n-3} \left[\frac{k}{2}\right] v_k. \tag{8}$$

Considering all the 3-valent, 4-valent and 5-valent vertices which are neighbors to a k-valent vertex, it follows by (2) that

$$e_{3,k} + e_{4,k} + e_{5,k} \leqq \left[\frac{k}{2}\right] v_k \qquad \text{for all } k, \tag{9}$$

hence

$$3v_3 + 4v_4 + 5v_5 - e_{5,n-5} = \sum_{k \geqq n-4} (e_{3,k} + e_{4,k} + e_{5,k}) \leqq \sum_{k \geqq n-4} \left[\frac{k}{2}\right] v_k. \tag{10}$$

By multiplying the inequalities (6), (8) and (10) by 5, 3 and 2, respectively, and adding them we get

$$\begin{aligned} &30v_3 + 20v_4 + 10v_5 - 2e_{5,n-5} - 3e_{4,n-4} - 5e_{3,n-3} \\ &\leqq 2\left[\frac{n-4}{2}\right] v_{n-4} + 5\left[\frac{n-3}{2}\right] v_{n-3} + 10 \sum_{k \geqq n-2} \left[\frac{k}{2}\right] v_k. \end{aligned} \tag{11}$$

Let e^* be defined by $e^* = 2e_{5,n-5} + 3e_{4,n-4} + 5e_{3,n-3}$; (1) multiplied by 10 and (11) yield

$$\begin{aligned} e^* \geqq\ & 120(1-g) + 10 \sum_{k=7}^{n-5} (k-6)v_k + \left(10(n-10) - 2\left[\frac{n-4}{2}\right]\right) v_{n-4} \\ &+ \left(10(n-9) - 5\left[\frac{n-3}{2}\right]\right) v_{n-3} + 10 \sum_{k \geqq n-2} \left(k - 6 - \left[\frac{k}{2}\right]\right) v_k, \end{aligned} \tag{12}$$

$$\begin{aligned} e^* \geqq\ & 120(1-g) + 10 \sum_{k=7}^{n-5} (k-6)v_k + (9n-95)v_{n-4} + (7\tfrac{1}{2}n - 82\tfrac{1}{2})v_{n-3} \\ &+ (5n-65)v_{n-2} + (5n-65)v_{n-1} + (5n-55)v_n + (5n-55)v_{n+1} + \cdots. \end{aligned} \tag{13}$$

If $g = 0$ and $n = 13$, then the coefficients of v_k in the right side of (13) are all nonnegative; therefore $e^* \geqq 120$, hence $w(G_0) \leqq 13$.

If $g = 1$ and $n = 15$, then coefficients of v_k in the right side of (13) are all positive, while $120(1-g) = 0$; therefore $e^* \geqq 1$, hence $w(G_1) \leqq 15$.

Examples show that indeed $w(G_0) = 13$ and $w(G_1) = 15$, see [6], [2], [3] and [4].

Suppose that $g \geqq 2$. Let V be a vertex of G of maximum degree t. By (3), $t \geqq (n+1)/2$.

Case 1. $(n+1)/2 \leqq t \leqq n-5$

If V has a neighbor W of degree s, then $w(VW) = t + s$, therefore $t + s \geqq n$, hence $s \geqq n - t \geqq 5$. If $s = 5$, then $w(VW) = n$ and therefore $w(G) = n$ and the proof is completed. Otherwise, $s \geqq 6$; $s \leqq t$ by the maximality of t, therefore

$s \leqq n-5$ and so each neighbor of V contributes to the right side of (13) at least $10(s-6)$, hence at least $10(n-t-6)$. By (3), at least $t-[t/2]$ of the neighbors of V have valences $\geqq (n+1)/2$, and $\leqq t \leqq n-5$. These neighbors contribute each at least $10((n+1)/2-6)$ to the right side of (13), while V itself contributes $10(t-6)$. It follows that

$$e^* \geqq 120(1-g)+10(t-6)+\left(t-\left[\frac{t}{2}\right]\right) 10\left(\frac{n+1}{2}-6\right)+\left[\frac{t}{2}\right] 10(n-t-6)$$

$$\geqq 120(1-g)+10(t-6)+(t/2)10\left(\frac{n+1}{2}-6\right)+\left(\frac{t-1}{2}\right) 10(n-t-6)$$

$$=-5t^2+t(7.5n-17.5)+62.5-120g.$$

The last expression in t has a maximum when $t=0.75n-1.75$, hence for a fixed g and a fixed n, the minimum value of the said expression where t varies so that $(n+1)/2 \leqq t \leqq n-6$ occurs for $t=(n+1)/2$, hence we get that

$$e^* \geqq -5\left(\frac{n+1}{2}\right)^2+\left(\frac{n+1}{2}\right)(7.5n-17.5)+62.5-120g$$

$$=2.5(n^2-3n+21-48g).$$

It follows that $e^* \geqq 1$ if $n^2-3n+21-48g>0$, i.e. if n is greater than the largest of the two roots of $n^2-3n+21-48g=0$, hence if $n > 1\frac{1}{2}+\sqrt{48g-18.75}$, hence if $n>6+\sqrt{48g+1}$, therefore if $n=n(g)$.

Case 2. $t=n-4$

In this case the vertex V contributes $9n-95$; by (3), at least $t-[t/2]= n-4-(n-5)/2=(n-3)/2$ of the neighbors of V have valences at least $(n+1)/2$ and at most t, hence they contribute each at least $10((n+1)/2-6)$, and so (13) yields

$$e^* \geqq 120(1-g)+9n-95+\left(\frac{n-3}{2}\right) 10\left(\frac{n+1}{2}-6\right)$$

$$=2.5n^2-26n+107.5-120g.$$

It follows as in the previous case that $e^* \geqq 1$ if $n>5.2+\sqrt{48g-15.96}$, hence $e^* \geqq 1$ if $n=n(g)$.

Case 3. $t=n-3$

The vertex V contributes $7\frac{1}{2}n-82\frac{1}{2}$ and $t-[t/2]=(n-3)/2$ of the neighbors of V have valences at least $(n+1)/2$ and at most t, hence they contribute each at least $10((n+1)/2-6)$, and so (13) yields

$$e^* \geqq 120(1-g)+7.5n-82.5+\left(\frac{n-3}{2}\right)10\left(\frac{n+1}{2}-6\right)$$

$$=2.5n^2-27.5n+120-120g.$$

It follows that $e^* \geqq 1$ if $n > 5.5+\sqrt{48g-17.75}$, hence if $n = n(g)$.

Case 4. $t \geqq n-2$

The vertex V contributes at least $5n-65$, and by (3) it has at least $t-[t/2] \geqq n-2-(n-3)/2=(n-1)/2$ of its neighbors contributing each at least $5n-65$, therefore by (13) it follows that

$$e^* \geqq 120(1-g)+5n-65+\left(\frac{n-1}{2}\right)(5n-65)$$

$$=2.5n^2-30n+87.5-120g;$$

therefore $e^* \geqq 1$ if $n > 6+\sqrt{48g+1}$, hence if $n = n(g)$.

Therefore if $n = n(g)$ then $e^* \geqq 1$, implying that $w(G)=n$, and the proof of Theorem 1 is complete.

Proof of Theorem 2. Let $g \geqq 2$ be such that $48g+1$ is a complete square, say $48g+1=(2m+1)^2$, and let G be a graph in G_g such that $w(G)=n(g)$. Hence $n=8+(2m+1)=2m+9$. It follows from (4), (7) and (9), which are applicable to our graph G and the same notation for the weight n, that

$$3v_3=\sum_{k\geqq n-3} e_{3,k} \leqq \sum_{k\geqq n-3}\left[\frac{k}{2}\right]v_k, \tag{14}$$

$$3v_3+4v_4=\sum_{k\geqq n-4}(e_{3,k}+e_{4,k}) \leqq \sum_{k\geqq n-4}\left[\frac{k}{2}\right]v_k, \tag{15}$$

$$3v_3+4v_4+5v_5=\sum_{k\geqq n-5}(e_{3,k}+e_{4,k}+e_{5,k}) \leqq \sum_{k\geqq n-5}\left[\frac{k}{2}\right]v_k. \tag{16}$$

It follows as in getting (12) from (6), (8) and (10), that

$$30v_3+20v_4+10v_5 \leqq 10\sum_{k\geqq n-3}\left[\frac{k}{2}\right]v_k+5\left[\frac{n-4}{2}\right]v_{n-4}+2\left[\frac{n-5}{2}\right]v_{n-5},$$

and by using (1) it follows that

$$\begin{aligned}120(g-1) \geqq\ & 10\sum_{k=7}^{n-6}(k-6)v_k+(9n-105)v_{n-5}+(7.5n-92.5)v_{n-4}+(5n-75)v_{n-3}\\ &+(5n-65)v_{n-2}+(5n-65)v_{n-1}+(5n-55)v_n+\cdots.\end{aligned} \tag{17}$$

By the proof of Theorem 1 it follows that $e^* \geqq 1$, hence $v_{n-3} + v_{n-4} + v_{n-5} \geqq 1$. Suppose $v_{n-3} = 0$. If $v_{n-4} \geqq 1$, then a $(n-4)$-valent vertex contributes $7.5n - 92.5$ to the right side of (17), and it has by (3) at least $n - 4 - [(n-4)/2] = (n-3)/2$ neighbors of valences $\geqq (n+1)/2$, contributing each at least $5n - 65$; therefore (17) implies

$$120(g-1) \geqq 7.5n - 92.5 + \frac{n-3}{2}(5n-65).$$

Since $48g + 1 = (2m+1)^2$ and $n = 2m + 9$, easy computations show that $m \leqq -7/3$, which is impossible. Therefore the supposition $v_{n-3} = 0$ leads to $v_{n-4} = 0$; hence $v_{n-5} \geqq 1$. In this case a $(n-5)$-valent vertex contributes $9n - 105$ and by (3) it has $(n-5)/2$ neighbors of valences $\geqq (n+1)/2$, contributing each at least $5n - 65$; therefore by (17) it follows that

$$120(g-1) \geqq 9n - 105 + \frac{n-5}{2}(5n-65),$$

which leads to $m \leqq -7$, impossible. Therefore it follows that $v_{n-3} \geqq 1$. A $(n-3)$-valent vertex contributes $5n - 75$, and it has by (3) $\geqq (n-3)/2$ neighbors of valences $\geqq (n+1)/2$, contributing each at least $5n - 75$; therefore

$$120(g-1) \geqq \left(1 + \frac{n-3}{2}\right)(5n-75). \tag{18}$$

In this case both of the two sides of the inequality equal $10(m^2 + m - 12)$, hence equality holds in (18); therefore

$$v_{n-3} = 1 + \frac{n-3}{2} = \frac{n-1}{2} = m + 4,$$

and as no other contributions are possible in the right side of (17), it follows that $v_k = 0$ for all k, $7 \leqq k \neq n-3$.

It follows by (16) that

$$3v_3 + 4v_4 + 5v_5 \leqq \left[\frac{n-3}{2}\right] v_{n-3} = m^2 + 7m + 12,$$

and (1) implies that

$$3v_3 + 2v_4 + v_5 = 12(1-g) + (n-3-6)\frac{n-1}{2} = m^2 + 7m + 12;$$

therefore $v_4 = v_5 = 0$ and $v_3 = \frac{1}{3}(m^2 + 7m + 12)$. In addition, it follows from the equality which was derived from (16), as well as from the equality that holds for the corresponding inequality (14), that every $(n-3)$-valent vertex has half of its neighbors as 3-valent vertices, in alternating order around it; therefore it follows that G can have no 6-valent vertices, i.e., $v_6 = 0$. By deleting all the 3-valent vertices from G we get a graph which triangulates S_g, has $m+4$ vertices which have valences $=(n-3)/2 = m+3$, hence it must be the complete graph K_{m+4} on $m+4$ vertices. Such a triangulation of S_g is known, by the famous Heawood–Youngs–Ringel Color Map Theorem, see [8].

This completes the proof of Theorem 2.

Proof of Theorem 3. Let G be in G_g such that $w(G) = n(g)$; and suppose G has a vertex of valency $\geqq n+2$. This vertex contributes at least $5n-45$ and it has at least $n+2-[(n+2)/2] = \frac{1}{2}(n+3)$ neighbors of valences $\geqq \frac{1}{2}(n+1)$, each of which contributes at least $5n-75$; therefore by (17)

$$120(g-1) \geqq 5n - 45 + \tfrac{1}{2}(n+3)(5n-75),$$

implying that $n \leqq 5 + \sqrt{48g+40}$. A quick check reveals that this contradicts the choice of n for all $g \geqq 1$, hence C has no vertices of valences $\geqq n+2$, and the proof is complete.

Improving Theorem 3 for many values of g, we have the following

Theorem 4. *There exists a set N_1 of integers, having density* 1, *such that if $G \in G_g$, $g \in N_1$ and $w(G) = n(g)$, then the maximum degree of G is at most $n-1$.*

Theorem 5. *There exists a set N_2 of integers, having density $\frac{1}{2}$, such that if $G \in G_g$, $g \in N_2$ and $w(G) = n(g)$, then the maximum degree of G is at most $n-3$.*

Proof of Theorem 4. If $G \in G_g$ and $w(G) = n(g)$, and if G has a vertex of valence $\geqq n$, then by (17) it follows that

$$120(g-1) \geqq 5n - 55 + \frac{n+1}{2}(5n-75),$$

implying that $n \leqq 6 + \sqrt{48g+25}$. This means that there should be an (odd) integer in the segment $(6+\sqrt{48g+1}, 6+\sqrt{48g+25}]$, or equivalently, that there should be a complete square in the segment $(48g+1, 48g+25]$. The set of values of g for which this holds has density zero, hence in its complement N_1 it does not happen, hence the maximum degree is then at most $n-1$.

Proof of Theorem 5. In a similar way, if $G \in G_g$ is such that $w(G) = n(g)$ and G has a vertex of valence $\geqq n-2$, then by (17) it follows that

$$120(g-1) \geqq 5n - 65 + \frac{n-1}{2}(5n-75),$$

implying that $n \leqq 7 + \sqrt{48g+12}$. This means that there should be an odd integer in the segment $(6+\sqrt{48g+1}, 7+\sqrt{48g+12}]$; these segments, for large values of g, have length decreasing to 1 and they contain an odd integer with probability $\frac{1}{2}$, hence if N_2 is defined as the set of those integers g for which the segments $(6+\sqrt{48g+1}, 7+\sqrt{48g+12}]$ contain no odd integers, then the said graph G has maximum valence which is at most $n-3$, and the density of N_2 is $\frac{1}{2}$.

Using the same method as in the proof of Theorem 3, we get the following result, in which the maximum valence ∇ and the weight w of a graph in G_g, are related:

THEOREM 6. *If G is a graph in $G_g, g \geqq 2$ and $w(G) = w \geqq 16$, and if ∇ denotes the maximum valence of the vertices of G, then the following inequalities hold, depending on the parity of w and ∇:*

	∇ even	∇ odd
w even	$\nabla \leqq 24\frac{2g-1}{w-12}$	$\nabla \leqq 24\frac{2g-1}{w-12} - 1$
w odd	$\nabla \leqq 24\frac{2g-1}{w-13}$	$\nabla \leqq 24\frac{2g-1}{w-13} - 1$

The proof of Theorem 6 requires the development of the analogue of (17) for even values of weights; therefore we omit the proof.

Concerning multigraphs and pseudographs on S_g, we have the following

THEOREM 7. *For every $g, g \geqq 1$, $w(MG_g) = 8g+7$; for every $g, g \geqq 2$, there exists a unique member H of MG_g such that $w(H) = 8g+7$.*

THEOREM 8. *For every $g, g \geqq 1$, $w(PG_g) = 24g-9$; for every $g, g \geqq 2$, there exists a unique member P of PG_g such that $w(P) = 24g-9$.*

PROOF OF THEOREM 7. Suppose a multigraph G in MG_g, $g \geqq 1$, is such that $w(G) \geqq 8g+7$; as (2) holds for G ((3) might not be true for a multigraph), it follows that (13) holds with $n = 8g+7$.

It is sufficient to show that G has at least three vertices of valences $\geqq (n+1)/2$, since then they contribute each $5n-65 = 40g-30$ to the right side of (13), implying that $e^* \geqq 120(1-g) + 3(40g-30) = 30$, hence $w(G) = 8g+7$.

A face of G has three different vertices, by the definition of MG_g; if all the three vertices of some face have valences $\geqq (n+1)/2$, then $e^* \geqq 30$ and

$w(G)=8g+7$. Otherwise, every triangle of G has just two vertices of valences $\geqq (n+1)/2$, and G has only two such vertices; if we denote by X and Y the two vertices of G of valences x and y, respectively, where $x, y \geqq (n+1)/2$, and if $Z_1, \cdots, Z_p$ are the other vertices of G, of valences $z_1, \cdots, z_p$, respectively, where $z_i \leqq \frac{1}{2}(n-1)$ for all i, $1 \leqq i \leqq p$, all the triangles of G are of the form XYZ_i. It follows that the vertices neighboring X are, in cyclic order, $Y, Z_{i,1}, Y, Z_{i,2}, \cdots, Y$; around the vertex Y they appear in the following cyclic order: $X, Z_{j,1}, X, Z_{j,2}, \cdots, X$, while around any of the Z_i vertices the neighbors are: $X, Y, X, Y, \cdots, X$. It follows that all z_i are even and $z_i \geqq 4$, and counting all the edges of the form XZ_i yields $x/2 = \Sigma_i (z_i/2)$, while counting all the edges of the form YZ_i leads to the equality $y/2 = \Sigma_i (z_i/2)$, therefore $x = y = \Sigma_i z_i$.

By (1) it follows that $0 = 12(1-g) + 2(x-6) + \Sigma_{i \geqq 1}(z_i - 6)$, hence

$$0 = 12 - 12g + 2x - 12 + \sum_{i \geqq 1} z_i - 6p = -12g + 2x + x - 6p,$$

implying that $x = 4g + 2p$.

As $x = \Sigma_{i \geqq 1} z_i$, it follows that $\min_i z_i \leqq x/p$, since x/p is the average value of the z_i. It follows that

$$x + \min_i z_i \leqq 4g + 2p + x/p = (4g+2p)(1+1/p);$$

since $z_i \geqq 4$ for all i, it follows that $4g + 2p = x = \Sigma_{i \geqq 1} z_i \geqq 4p$, hence $4g \geqq 2p$, or $p \leqq 2g$. Therefore

$$w(G) = x + \min_i z_i \leqq (4g+2p)(1+1/p)$$

$$\leqq \max\{(4g+2p)(1+1/p) \mid 1 \leqq p \leqq 2g\} = 8g+4$$

(as can be easily obtained by calculus, where the maximum is obtained for $p=1$). This contradicts the assumption on G that $w(G) \geqq 8g+7$.

It follows that some face of G has all of its three vertices of valence $(n+1)/2$, and therefore $w(G) = 8g+7$.

Suppose that for some multigraph H in MG_g, $g \geqq 2$, $w(H) = 8g+7$. By the previous part of the proof it was shown that H has at least three vertices of valence $\geqq (n+1)/2$, and using (17) we get

$$120(g-1) \geqq 3(5n-75) = 3(5(8g+7)-75) = 120g - 120,$$

therefore equality holds and hence there are no other contributions in the right side of (17), implying that $v_k = 0$ for all k, $7 \leqq k \neq n-3$.

If $g = 1$, then $n = 15$ and $5n - 75 = 0$, hence the only conclusion for $v_{n-3} = v_{12}$ is that $v_{12} = 3$; for examples of multigraphs in MG_1 for which $w = 15$, see [4].

Suppose that $g \geqq 2$, $5(8g+7)-75>0$, hence it follows that $v_{n-3} = 3$, and $v_k = 0$ for all k, $7 \leqq k \neq n-3$.

By (1) it follows that $3v_3 + 2v_4 + v_5 = 12(1-g)+3(n-3-6) = 12g+6$; by (16) it follows that

$$3v_3 + 4v_4 + 5v_5 \leqq 3\left[\frac{n-3}{2}\right] = \frac{3(8g+7-3)}{2} = 12g+6,$$

hence it follows that $v_4 = v_5 = 0$. Therefore $v_3 = 4g+2$, and $v_{8g+4} = 3$; no 6-valent vertex is a neighbor of a 3-valent vertex, and all three neighbors of a 3-valent vertex must be different, therefore every 3-valent vertex is a neighbor to each one of the three $(8g+4)$-valent vertices. It follows that the 3-valent vertices appear cyclically in an alternating fashion around every $(8g+4)$-valent vertex; therefore H contains no 6-valent vertices.

To construct this unique H, let H_1 be the tessellation of the torus shown in Fig. 1; H_1 has three vertices of valence 6. To get H_g, assume H_{g-1} has been

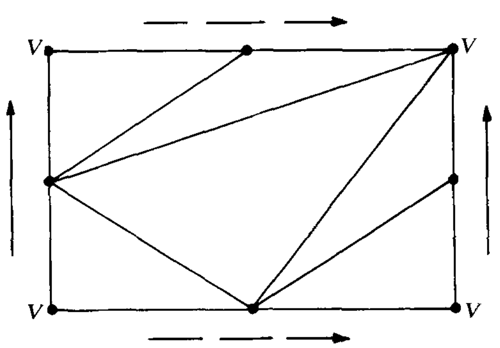

Fig. 1.

constructed as an element of MG_{g-1} having just three vertices, which have valence $4(g-1)+2$; H_g is obtained from H_{g-1} and H_1 by deleting one triangular face from each, and identify the boundaries properly along the three edges. H_g is a member of MG_g and it has just three vertices which have valency $4g+2$. The multigraph H is obtained from H_g by splitting each triangular face of H_g into three triangles meeting at a 3-valent vertex (the corresponding Kleetope over H_g). The three $(4g+2)$-valent vertices of H_g are converted into three $(8g+4)$-valent vertices in H, and the weights of the edges of H are either $8g+7$, or $16g+8$, hence $w(H) = 8g+7$.

It follows that for every g, $g \geqq 1$, $w(MG_g) = 8g + 7$, and that there exists a unique multigraph H in MG_g, for all $g \geqq 2$, such that $w(H) = 8g + 7$; this completes the proof of Theorem 7.

PROOF OF THEOREM 8. Suppose P is a pseudograph in PG_g, $g \geqq 1$, such that $w(P) \geqq 24g - 9$; as (2) holds for P, it follows that (13) holds, with $n = 24g - 9$.

Clearly P has at least one vertex of valence $\geqq (n+1)/2$, which contributes at least $5n - 65$ to the right side of (13), therefore

$$e^* \geqq 120(1-g) + 5n - 65 = 120(1-g) + 5(24g-9) - 65 = 10,$$

therefore $e^* \geqq 10$, and hence $w(P) = 24g - 9$.

Suppose $\bar{P}$ is in PG_g, for $g \geqq 2$, such that $w(\bar{P}) = 24g - 9$. (17) holds for $\bar{P}$ and since $\bar{P}$ has at least one vertex of valence $\geqq (n+1)/2$, it follows that

$$120(g-1) \geqq 5n - 75 = 5(24g-9) - 75 = 120(g-1),$$

therefore equality holds and $v_{n-3} = v_{24g-12} = 1$ and $v_k = 0$ for all k, $7 \leqq k \neq 24g - 12$.

By (1) it follows that $3v_3 + 2v_4 + v_5 = 12(1-g) + 24g - 12 - 6 = 12g - 6$; by (16) it follows that

$$3v_3 + 4v_4 + 5v_5 \leqq \left[\frac{n-3}{2}\right] = \frac{24g-12}{2} = 12g - 6,$$

hence it follows that $v_4 = v_5 = 0$, and that $v_3 = 4g - 2$. A 6-valent vertex is not a neighbor of any 3-valent vertex, and it follows as in the proof of the previous theorem that $v_6 = 0$.

To construct the unique P in PG_g, $g \geqq 2$, for which $w(P) = 24g - 9$, start with $2g - 1$ loops on the sphere, as shown in Fig. 2, where all the loops have a common vertex V and they determine $2g$ regions, of which $2(g-1)$ are digons and the other two are monogons. Replace the interiors of a pair of nonadjacent digons by the handle, shown in Fig 3, and where the four vertices V^i, $1 \leqq i \leqq 4$, in the handle are identified with the vertex V on the sphere; this operation is performed $g - 1$ times using all of the digons. One additional operation is required, in which the interiors of the monogons are deleted and a handle is inserted, as shown in Fig. 4, where the two vertices V^1 and V^2 are identified with the vertex V on the sphere. The result is a pseudograph on S_g, $g \geqq 2$, having just one vertex, of valence $2(2g-1) + 8(g-1) + 4 = 12g - 6$. The Kleetope over it is a pseudograph in S_g in which 3-valent vertices are neighbors to the only other vertex of valence $2(12g-6) = 24g - 12$, therefore its weight equals $24g - 9$.

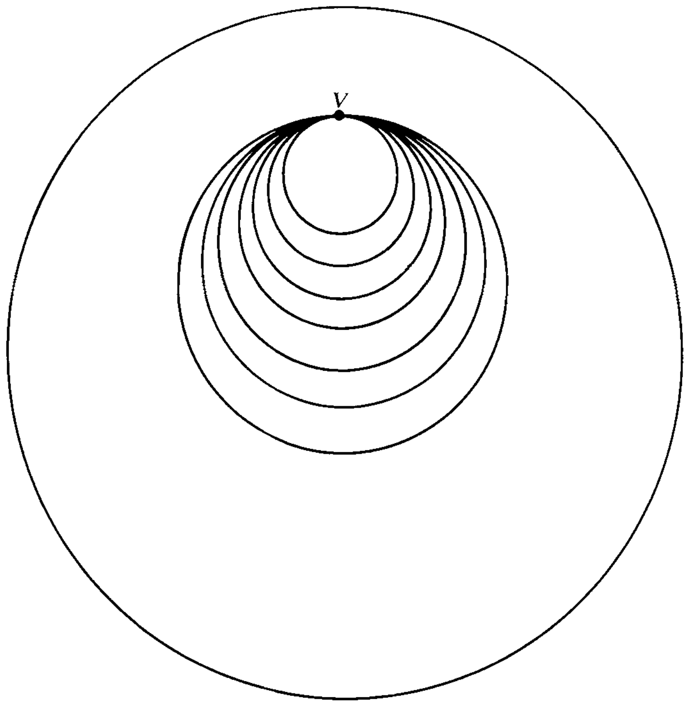

Fig. 2.

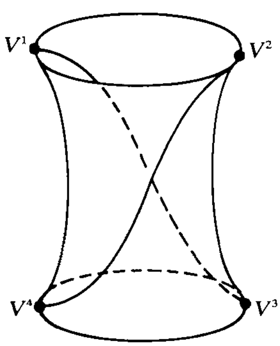

Fig. 3.

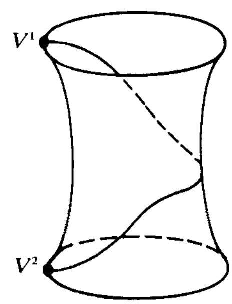

Fig. 4.

This completes the proof of Theorem 8.

As a corollary to the proof of Theorem 1, we have

Corollary 1. *If a multigraph or a pseudograph in* $MG_g \cup PG_g$, $g \geqq 1$, *is such that a vertex of maximum degree* t *has* t *different neighbors, then* $w(G) \leqq n(g)$.

PROOF. By use of (3); we omit the obvious details.

For large graphs we have the following

THEOREM 9. *If* $\{G^m\}_{m=1}^{\infty}$ *is a sequence of pseudographs such that* $V(G^m)\to\infty$ *as* $m\to\infty$, *and where for a fixed* g_0, G^m *is in* $PG_{g(m)}$ *with* $g(m)\leqq g_0$ *for all* m, *then there exists an* M *such that* $w(G^m)\leqq 15$ *holds for all* m, $m\geqq M$.

PROOF OF THEOREM 9. It is clearly sufficient to prove the assertion for sequences $\{G^m\}_{m=1}^{\infty}$ satisfying, in addition, the condition $w(G^m)\geqq 15$ (take the subsequence, obtained by deleting all G^m for which $w(G^m)\leqq 14$). Applying (13) with $n=15$ for all G^m, we observe that all the coefficients of $v_k(G^m)$ are positive.

Let $\{G^{m'}\}_{m'\geqq 1}$ be the subcollection (finite or infinite) of $\{G^m\}_{m=1}^{\infty}$ of all those graphs $G^{m'}$ for which the expression of the right side of (13) is negative or zero, i.e., for which

$$10\sum_{k=7}^{10}(k-6)v_k(G^{m'})+50v_{11}(G^{m'})+30v_{12}(G^{m'})+10v_{13}(G^{m'})+10v_{14}(G^{m'})+\cdots$$

$$\leqq 120(g(m')-1)\leqq 120(g_0-1).$$

It follows that for some L_1 and L_2, $v_k(G^{m'})\leqq L_1$ holds for all k, $7\leqq k\leqq L_2$ and $v_k(G^{m'})=0$ for all k, $k>L_2$.

By (1) it follows that $v_3(G^{m'})$, $v_4(G^{m'})$ and $v_5(G^{m'})$ are bounded as well, i.e., there exist constants L_3 and L_4 such that $v_k(G^{m'})\leqq L_3$ holds for all k, $3\leqq k\leqq L_4$ and $k\neq 6$, and $v_k(G^{m'})=0$ for all k, $k>L_4$.

If the collection $\{G^{m'}\}$ is infinite, then since $V(G^m)\to\infty$ as $m\to\infty$, it follows that, in addition, $v_6(G^{m'})\to\infty$ as $m'\to\infty$; therefore if m' is big enough, $G^{m'}$ must have 6-valent vertices neighboring 6-valent vertices, i.e., $e_{6,6}(G^{m'})\geqq 1$, implying that $w(G^{m'})\leqq 12$, in contradiction to the assumption on $G^{m'}$.

It follows therefore that the collection of $G^{m'}$ is finite, hence for some M, $m'<M$ for all m'; equivalently, $e^*(G^m)\geqq 1$ for all m, $m>M$, i.e. $w(G^m)\leqq 15$ holds for all m, $m\geqq M$.

This completes the proof of Theorem 9.

THEOREM 10. *For every* g, $g\geqq 0$, *there exists a constant* $C=C(g)$ *such that if a pseudograph* G *is in* PG_g *and* $V(G)\geqq C$, *then* $w(G)\leqq 15$.

PROOF OF THEOREM 10. The assertion is clearly true for $g=0,1$. If the assertion is false for some g, $g\geqq 2$, then there exists a sequence $\{G^m\}_{m=1}^{\infty}$ of graphs in G_g such that $V(G^m) \to \infty$ as $m \to \infty$, and where $w(G^m) > 15$ for all m; this contradicts Theorem 9.

In a similar way we prove the following

THEOREM 11. *For every integer g, $g \geqq 0$, and for every constant C there exists a constant $C^* = C^*(g, C)$, such that if a pseudograph G is in PG_g and $V(G) \geqq C^*$, then*

(i) $w(G) \leqq 14$, *if in addition* $v_{12}(G) \leqq C$;
(ii) $w(G) \leqq 13$, *if in addition* $v_{11}(G)$ & $v_{12}(G) \leqq C$;
(iii) $w(G) \leqq 12$, *if in addition* $v_{10}(G)$, $v_{11}(G)$ & $v_{12}(G) \leqq C$.

The proof of Theorem 11 is omitted, being very similar to the proofs of the previous theorems; merely observe that the coefficients of v_k in (13), or in its analogue for even weights, which are either negative or zero, do multiply bounded values of $v_k - s$.

REMARK. Our bounds in Theorem 1 agree with the conjectured bound of $13 + 2g$ for all g, $0 \leqq g \leqq 6$, and are smaller than $13 + 2g$ for all g, $g \geqq 7$; all the bounds in Theorems 1, 5 and 6, for $g = 1$, are equal to 15.

Let G_g^*, G_g^{**} and G_g^{***} be defined by

$$G_g^* = \{G \mid G \in G_g \text{ and } V_3(G) = 0\},$$

$$G_g^{**} = \{G \mid G \in G_g \text{ and } V_3(G) = V_4(G) = 0\},$$

$$G_g^{***} = \{G \mid G \in G_g \text{ and } V_3(G) = V_4(G) = V_5(G) = 0\}.$$

Kotzig [6] proved that $w(G_0^*) = w(G_0^{**}) = 11$, and Grünbaum and Shephard [4] proved that $w(G_1^*) = w(G_1^{**}) = 12$. We have the following

THEOREM 12. *For every g, $g \geqq 1$, $w(G_g^{***}) \leqq n(g) - 3$; for every g, $g \geqq 2$, making $48g + 1$ a complete square, there exists a unique graph G in G_g^{***} such that $w(G) = n(g) - 3$.*

THEOREM 13. *For every g, $g \geqq 2$, $w(G_g^*)$ is at most the least odd integer greater than $5.5 + \sqrt{48g - 17.75}$.*

THEOREM 14. *For every g, $g \geqq 2$, $w(G_g^{**})$ is at most the minimum of the least odd integer greater than $6 + \sqrt{48g - 19}$ and the least even integer greater than $6.2 + \sqrt{48g + 4.84}$.*

PROOF OF THEOREM 12. If $G \in G_g^{***}$, then by (1), $12(g - 1) = \Sigma_{k \geqq 7} (k - 6) v_k$ and the bound on the weight of G follows easily from (3).

If for $g \geqq 2$, $48g+1=(2m+1)^2$ and G in G_g^{***} is such that $w(G)=n(g)-3=n-3$, then $n-3=(8+2m+1)-3=2m+6$ and $12g=m^2+m$. By (3), G has at least one vertex of valence $\geqq (n-3)/2=m+3$, and at least $m+3$ of its neighbors are of valence $\geqq m+3$; thus G has at least $m+4$ vertices of valences $\geqq m+3$. Therefore

$$12(g-1) \geqq (m+4)(m+3-6)=(m+4)(m-3)=m^2+m-12=12g-12,$$

since $m^2+m=12g$; thus equality holds, implying that $v_k=0$ for all k, $7 \leqq k \neq m+3$. No neighbor of a $(m+3)$-valent vertex is 6-valent, because of the restriction on the weight of G, therefore $v_6=0$. It follows that G must be the complete graph K_{m+4} on $m+4$ vertices, known to triangulate S_g by the Heawood–Youngs–Ringel map coloring theorem. This completes the proof of Theorem 12.

The proofs of Theorems 13 and 14 are omitted, since they are the straight analogues of the proof of Theorem 1, in which the corresponding inequalities are obtained for both odd and even weights (in the case of Theorem 14).

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